COMPLETE VERTICAL GRAPHS WITH CONSTANT MEAN CURVATURE IN SEMI-RIEMANNIAN WARPED PRODUCTS

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Abstract. In this paper we study complete vertical graphs of constant mean curvature in the Hyperbolic and Steady State spaces. We first derive suitable formulas for the Laplacians of the height function and of a support-like function naturally attached to the graph; then, under appropriate restrictions on the values of the mean curvature and the growth of the height function, we obtain necessary conditions for the existence of such a graph. In the two-dimensional case we apply this analytical framework to state and prove Bernstein-type results in each of these ambient spaces.

1. Introduction

This paper deals with complete non-compact constant mean curvature graphs over a horosphere of the Hyperbolic space, as well as over horizontal hyperplanes (*slices*) in the Steady State space. In connection with our work, L. Alías and M. Dajczer (cf. [2]) studied properly immersed complete surfaces of the $3-$dimensional Hyperbolic space contained between two horospheres, obtaining a Bernstein-type result for the case of constant mean curvature between -1 and 1. In de Sitter space, K. Akutagawa (cf. [5]) proved that complete spacelike hypersurfaces having constant mean curvature in a specific interval of the real line are totally umbilical. Also for de Sitter space, among other interesting results S. Montiel (cf. [14]) proves that, under an appropriate restriction on their Hyperbolic Gauss map, complete spacelike Hypersurfaces of constant mean curvature greater than or equal to 1 must actually have mean curvature 1.

For the Lorentz case, our motivation to restrict attention to the Steady State space comes from the fact that there exists a natural duality between the Gauss maps of Riemannian hypersurfaces of this space and those of the Hyperbolic space, provided we model these as hyperquadrics of the Lorentz-Minkowski space (cf. section 5). Besides, in physical context the Steady State space appears naturally as an exact solution for the Einstein equations, being a cosmological model where matter is supposed to travel along geodesics normal to horizontal hyperplanes; these, in turn, serve as the initial data for the Cauchy problem associated to those equations (cf. [8], chapter 5).

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In this work we model both our ambient spaces as semi-Riemannian warped products to obtain necessary conditions for the existence of the types of graphs mentioned in the beginning of this introduction. More precisely, under appropriate restrictions on the values of the mean curvature and the growth of the height function of these graphs, we actually prove that the mean curvature has to be identically 1 (cf. Theorem 4.1 and Theorem 5.1). We also prove (under a slightly stronger hypothesis in the Hyperbolic case) that the scalar curvature of our graphs cannot be globally bounded away from zero in a certain sense. The analytical framework we use to prove the above-mentioned results consists of the generalized maximum principle of Omori and Yau. Specifically, we apply lemma 3 of [5] on nonnegative solutions to the partial differential inequality $\Delta g \geq ag^2$ (a being a positive real constant) to a carefully chosen combination of functions naturally attached to our immersions.

In dimension 2, for complete surfaces of nonnegative Gaussian curvature, we are able to obtain Bernstein-type theorems related to our previous general results by using the fact that those surfaces are parabolic in the sense of Riemann surfaces (cf. [9]). Indeed, if the size of the gradient of the height function of the graph is suitably bounded, then the graph has to be a horosphere in the $3-$dimensional Hyperbolic space (cf. Theorem 5.2), or a horizontal plane in the $3-$dimensional Steady State space (cf. Theorem 4.5).

This paper is organized in the following manner: in section 2 we discuss general semi-Riemannian manifolds furnished with conformal vector fields, and derive a formula for the Laplacian of a support-like function associated to an oriented Riemannian hypersurface of such an ambient space. Section 3 recasts the result of the previous one in the particular context of semi-Riemannian warped products with Riemannian fiber; we also compute the Laplacian of a general height function and close the section by defining the objects of our main interest, namely, vertical graphs over fibers of such an warped product. Finally, sections 4 and 5 are respectively devoted to applications of this general picture to the special cases of the Steady State space and the Hyperbolic space.

Finally, it was communicated to us by professor L.J. Alías that our results about spacelike surfaces in the 3-dimensional steady state space (Section 4) are somewhat related to a work in progress due to him and A. L. Albujer (cf. [1]).

2. Conformal vector fields

Let $\overline{M}^{n+1}$ be a connected semi-Riemannian manifold with metric $\overline{g} = \langle\ ,\ \rangle$ of index $\nu \leq 1$, and semi-Riemannian connection $\overline{\nabla}$. For a vector field $X \in \mathcal{X}(\overline{M})$, let $\epsilon(X) = \langle X, X\rangle$; X is said to be a *unit* vector field if $\epsilon(X) = \pm 1$, *timelike* if $\epsilon(X) = -1$.

A vector field V on $\overline{M}^{n+1}$ is said to be *conformal* if

$$\mathcal{L}_V\langle\ ,\ \rangle = 2\phi\langle\ ,\ \rangle \tag{2.1}$$

for some function $\phi \in C^\infty(\overline{M})$, where $\mathcal{L}$ stands for the Lie derivative of the metric of $\overline{M}$. The function ϕ is called the *conformal factor* of V.

Since $\mathcal{L}_V(X) = [V, X]$ for all $X \in \mathcal{X}(\overline{M})$, it follows from the tensorial character of $\mathcal{L}_V$ that $V \in \mathcal{X}(\overline{M})$ is conformal if and only if

$$\langle\overline{\nabla}_X V, Y\rangle + \langle X, \overline{\nabla}_Y V\rangle = 2\phi\langle X, Y\rangle, \tag{2.2}$$

for all $X, Y \in \mathcal{X}(\overline{M})$. In particular, V is a Killing vector field relatively to $\overline{g}$ if and only if $\phi \equiv 0$.

In all that follows, we consider *Riemannian immersions* $\psi : \Sigma^n \to \overline{M}^{n+1}$, namely, immersions from a connected, $n-$dimensional orientable differentiable manifold Σ into $\overline{M}$, such that the induced metric $g = \psi^*(\overline{g})$ turns Σ into a Riemannian manifold (in the Lorentz case $\nu = 1$, we refer to (Σ, g) as a *spacelike* hypersurface of $\overline{M}$), with Levi-Civita connection ∇. We orient Σ by the choice of a unit normal vector field N on it, let A denote the corresponding shape operator and $H = \epsilon(N)\,\mathrm{tr}(A)/n$ the corresponding mean curvature.

The following proposition appeared for the first time in [16], there in the Riemannian setting. In a joint work with A. B. Barros and A. Brasil (cf. [6]) the first author generalized it to the Lorentz setting. Here we present a unified version of it, together with a proof.

Proposition 2.1. *Let $\overline{M}^{n+1}$ be semi-Riemannian manifold furnished with a conformal vector field V with conformal factor $\phi : \overline{M}^{n+1} \to \mathbb{R}$, and $\psi : \Sigma^n \to \overline{M}^{n+1}$ a Riemannian immersion. If $\eta = \langle V, N\rangle$, then*

$$\Delta\eta = -\epsilon\, n\langle V, \nabla H\rangle - \epsilon\,\eta\left\{\overline{\mathrm{Ric}}(N,N) + |A|^2\right\} - n\left\{\epsilon\, H\phi + N(\phi)\right\}, \tag{2.3}$$

where $\epsilon = \epsilon(N)$, ∇H the gradient of H in the metric of Σ, $\overline{\mathrm{Ric}}$ is the Ricci tensor of $\overline{M}$ and $|A|$ is the Hilbert-Schmidt norm of A.

Proof. Fix $p \in \Sigma$ and let $\{e_k\}$ be an orthonormal moving frame on a neighborhood of p in Σ, geodesic at p. Extend the e_k to a neighborhood of p in $\overline{M}$, so that $(\overline{\nabla}_N e_k)(p) = 0$, and let

$$V = \sum_l^n \alpha_l e_l + \epsilon\,\eta N.$$

Then

$$\begin{aligned} \eta = \langle N, V\rangle \Rightarrow e_k(\eta) &= \langle \overline{\nabla}_{e_k} N, V\rangle + \langle N, \overline{\nabla}_{e_k} V\rangle \\ &= -\langle Ae_k, V\rangle + \langle N, \overline{\nabla}_{e_k} V\rangle, \end{aligned}$$

so that

$$\begin{aligned} \Delta\eta &= \sum_k e_k(e_k(\eta)) = -\sum_k e_k\langle Ae_k, V\rangle + \sum_k e_k\langle N, \overline{\nabla}_{e_k} V\rangle \\ &= -\sum_k \langle \overline{\nabla}_{e_k} Ae_k, V\rangle - 2\sum_k \langle Ae_k, \overline{\nabla}_{e_k} V\rangle + \sum_k \langle N, \overline{\nabla}_{e_k}\overline{\nabla}_{e_k} V\rangle. \end{aligned} \tag{2.4}$$

Now, differentiating $Ae_k = \sum_l h_{kl} e_l$ with respect to e_k, one gets at p

$$\begin{aligned} \sum_k \langle \overline{\nabla}_{e_k} Ae_k, V\rangle &= \sum_{k,l} e_k(h_{kl})\langle e_l, V\rangle + \sum_{k,l} h_{kl}\langle \overline{\nabla}_{e_k} e_l, V\rangle \\ &= \sum_{k,l} \alpha_l e_k(h_{kl}) + \epsilon\sum_{k,l} h_{kl}\langle \overline{\nabla}_{e_k} e_l, N\rangle\langle V, N\rangle \\ &= \sum_{k,l} \alpha_l e_k(h_{kl}) + \epsilon\sum_{k,l} h_{kl}^2\eta \\ &= \sum_{k,l} \alpha_l e_k(h_{kl}) + \epsilon\,\eta|A|^2. \end{aligned} \tag{2.5}$$

Asking further that $Ae_k = \lambda_k e_k$ at p (which is always possible), we have at p

$$\sum_k \langle Ae_k, \overline{\nabla}_{e_k} V\rangle = \sum_k \lambda_k \langle e_k, \overline{\nabla}_{e_k} V\rangle = \sum_k \lambda_k \phi = \epsilon\, nH\phi. \tag{2.6}$$

In order to compute the last summand of (2.4), note that the conformality of V gives

$$\langle \overline{\nabla}_N V, e_k\rangle + \langle N, \overline{\nabla}_{e_k} V\rangle = 0$$

for all k. Hence, differentiating the above relation in the direction of e_k, we get

$$\langle \overline{\nabla}_{e_k} \overline{\nabla}_N V, e_k\rangle + \langle \overline{\nabla}_N V, \overline{\nabla}_{e_k} e_k\rangle + \langle \overline{\nabla}_{e_k} N, \overline{\nabla}_{e_k} V\rangle + \langle N, \overline{\nabla}_{e_k}\overline{\nabla}_{e_k} V\rangle = 0.$$

However, at p one has

$$\begin{aligned} \langle \overline{\nabla}_N V, \overline{\nabla}_{e_k} e_k\rangle &= \epsilon\langle \overline{\nabla}_N V, \langle \overline{\nabla}_{e_k} e_k, N\rangle N\rangle = \epsilon\langle \overline{\nabla}_N V, \lambda_k N\rangle \\ &= \epsilon\lambda_k\phi\langle N, N\rangle = \lambda_k\phi \end{aligned}$$

and

$$\langle \overline{\nabla}_{e_k} N, \overline{\nabla}_{e_k} V\rangle = -\lambda_k\langle e_k, \overline{\nabla}_{e_k} V\rangle = -\lambda_k\phi,$$

so that

$$\langle \overline{\nabla}_{e_k} \overline{\nabla}_N V, e_k\rangle + \langle N, \overline{\nabla}_{e_k}\overline{\nabla}_{e_k} V\rangle = 0 \tag{2.7}$$

at p. On the other hand, since

$$[N, e_k](p) = (\overline{\nabla}_N e_k)(p) - (\overline{\nabla}_{e_k} N)(p) = \lambda_k e_k(p),$$

it follows from (2.7) that

$$\begin{aligned} \langle \overline{R}(N, e_k)V, e_k\rangle_p &= \langle \overline{\nabla}_{e_k}\overline{\nabla}_N V - \overline{\nabla}_N\overline{\nabla}_{e_k} V + \overline{\nabla}_{[N,e_k]} V, e_k\rangle_p \\ &= -\langle N, \overline{\nabla}_{e_k}\overline{\nabla}_{e_k} V\rangle_p - N\langle \overline{\nabla}_{e_k} V, e_k\rangle_p + \langle \overline{\nabla}_{\lambda_k e_k} V, e_k\rangle_p \\ &= -\langle N, \overline{\nabla}_{e_k}\overline{\nabla}_{e_k} V\rangle_p - N(\phi) + \lambda_k\phi, \end{aligned}$$

and hence

$$\sum_k \langle N, \overline{\nabla}_{e_k}\overline{\nabla}_{e_k} V\rangle_p = -nN(\phi) + \epsilon\, nH\phi - \overline{\mathrm{Ric}}(N, V)_p \tag{2.8}$$

Finally,

$$\begin{aligned} \overline{\mathrm{Ric}}(N, V) &= \sum_l \alpha_l \overline{\mathrm{Ric}}(N, e_l) + \epsilon\,\eta\overline{\mathrm{Ric}}(N, N) \\ &= \sum_{k,l} \alpha_l\langle \overline{R}(e_k, e_l)e_k, N\rangle + \epsilon\,\eta\overline{\mathrm{Ric}}(N, N), \end{aligned}$$

and

$$\begin{aligned} \langle \overline{R}(e_k, e_l)e_k, N\rangle_p &= \langle \overline{\nabla}_{e_l}\overline{\nabla}_{e_k} e_k - \overline{\nabla}_{e_k}\overline{\nabla}_{e_l} e_k, N\rangle_p \\ &= e_l\langle \overline{\nabla}_{e_k} e_k, N\rangle_p - \langle \overline{\nabla}_{e_k} e_k, \overline{\nabla}_{e_l} N\rangle_p - e_k\langle \overline{\nabla}_{e_l} e_k, N\rangle_p \\ &\quad + \langle \overline{\nabla}_{e_l} e_k, \overline{\nabla}_{e_k} N\rangle_p \\ &= -e_l\langle e_k, \overline{\nabla}_{e_k} N\rangle_p + e_k\langle e_k, \overline{\nabla}_{e_l} N\rangle_p \\ &= e_l(h_{kk}) - e_k(h_{kl}), \end{aligned}$$

so that

$$\overline{\mathrm{Ric}}(N, V)_p = \sum_{k,l} \alpha_l e_l(h_{kk}) - \sum_{k,l} \alpha_l e_k(h_{kl}) + \epsilon\,\eta\overline{\mathrm{Ric}}(N, N)_p,$$

and it follows from (2.8) that

$$\begin{aligned}\sum_k \langle N, \overline{\nabla}_{e_k}\overline{\nabla}_{e_k}V\rangle_p &= -nN(\phi) + \epsilon\, nH\phi - V^\top(\epsilon\, nH)\\ &\quad + \sum_{k,l}\alpha_l e_k(h_{kl}) - \epsilon\,\eta\overline{\mathrm{Ric}}(N,N).\end{aligned} \tag{2.9}$$

Substituting (2.5), (2.6) and (2.9) into (2.4), one gets the desired formula (2.3). □

3. Semi-Riemannian warped products

Let M^n be a connected, n-dimensional oriented Riemannian manifold, $I \subset \mathbb{R}$ an interval and $f : I \to \mathbb{R}$ a positive smooth function. In the product differentiable manifold $\overline{M}^{n+1} = I \times M^n$, let π_I and π_M denote the projections onto the I and M factors, respectively.

A particular class of semi-Riemannian manifolds having conformal fields is the one obtained by furnishing $\overline{M}$ with the metric

$$\langle v, w\rangle_p = \epsilon\langle(\pi_I)_*v, (\pi_I)_*w\rangle + f(p)^2\langle(\pi_M)_*v, (\pi_M)_*w\rangle,$$

where $\epsilon = -1$ or $\epsilon = 1$ for all $p \in \overline{M}$ and all $v, w \in T_p\overline{M}$. Indeed (cf. [12] and [13]), the vector field

$$V = (f \circ \pi_I)\partial_t$$

is conformal and closed (in the sense that its dual 1−form is closed), with conformal factor $\phi = f'$, where the prime denotes differentiation with respect to $t \in I$. Such a space is called a semi-Riemannian *warped product*, and in what follows we shall write $\overline{M}^{n+1} = \epsilon I \times_f M^n$ to denote it.

If $\psi : \Sigma^n \to \overline{M}^{n+1}$ is a Riemannian immersion, with Σ oriented by the unit vector field N, one obviously has $\epsilon = \epsilon(\partial_t) = \epsilon(N)$. The following result restates proposition 2.1 in this context, in the spirit of [4].

Proposition 3.1. *Let $\overline{M}^{n+1} = \epsilon\, I \times_f M^n$. In the notations of proposition 2.1, if Σ has constant mean curvature H, then*

$$\Delta\eta = -\epsilon\,\eta\left\{\mathrm{Ric}(N^\top, N^\top) + (n-1)(\log f)''(1 - \langle N, \partial_t\rangle^2) + |A|^2\right\} - \epsilon\, nHf'. \tag{3.1}$$

where Ric *denotes the Ricci tensor of M and $N^\top = (\pi_M)_*N$.*

Proof. First of all, $\eta = \langle V, N\rangle = f\langle N, \partial_t\rangle$, and it thus follows from (2.3) that

$$\Delta\eta = -\epsilon\,\eta\left\{\overline{\mathrm{Ric}}(N,N) + |A|^2\right\} - n\left\{\epsilon Hf' + N(f')\right\}.$$

Now, $N(f') = \epsilon\, f''\langle N, \partial_t\rangle = \epsilon\,(f''/f)\eta$. On the other hand, since $N = N^\top + \epsilon\langle N, \partial_t\rangle\partial_t$, it follows from corollary 7.43 of [15] that

$$\begin{aligned}\overline{\mathrm{Ric}}(N,N) &= \overline{\mathrm{Ric}}(N^\top, N^\top) + \langle N, \partial_t\rangle^2\overline{\mathrm{Ric}}(\partial_t, \partial_t)\\ &= \mathrm{Ric}(N^\top, N^\top) - \epsilon\langle N^\top, N^\top\rangle\left\{\frac{f''}{f} + (n-1)\frac{(f')^2}{f^2}\right\} - \frac{nf''}{f}\langle N, \partial_t\rangle^2\\ &= \mathrm{Ric}(N^\top, N^\top) - \left\{\frac{f''}{f} + (n-1)\frac{(f')^2}{f^2}\right\} - (n-1)\left(\frac{f'}{f}\right)'\langle N, \partial_t\rangle^2,\end{aligned}$$

where we used that $\langle N^\top, N^\top\rangle = \epsilon(1 - \langle N, \partial_t\rangle^2)$ in the last equality above.

$$\begin{aligned}\Delta\eta &= -\epsilon\,\eta\left\{\mathrm{Ric}(N^\top, N^\top) - \left\{\frac{f''}{f} + (n-1)\frac{(f')^2}{f^2}\right\} - (n-1)\left(\frac{f'}{f}\right)'\langle N, \partial_t\rangle^2\right\}\\ &\quad -\epsilon\,\eta|A|^2 - \epsilon\, n\left\{Hf' + \frac{f''}{f}\eta\right\}\\ &= -\epsilon\,\eta\left\{\mathrm{Ric}(N^\top, N^\top) + (n-1)(\log f)''(1 - \langle N, \partial_t\rangle^2) + |A|^2\right\} - \epsilon\, nHf'.\end{aligned}$$

□

If $\psi : \Sigma^n \to \overline{M}^{n+1}$ is a Riemannian immersion as above, we let $h = \pi_{I_{|\Sigma}} : \Sigma \to I$ denote the height function of Σ with respect to the unit vector field ∂_t. As far as we know, the following proposition appeared for the first time in [4], as a special case of lemma 4.1; here we present a direct proof of the particular case which is needed for the applications we have in mind. We would like to thank professors L. Alías and A. G. Colares for having showed us the above mentioned preprint, which enabled us to set this result.

Proposition 3.2. *In the above notation,*

$$\Delta h = (\log f)'(h)\{\epsilon\, n - |\nabla h|^2\} + \epsilon\, nH\langle N, \partial_t\rangle, \tag{3.2}$$

where H denotes the mean curvature of Σ with respec to N.

Proof. Since $h = \pi_{I_{|\Sigma}}$, one has

$$\begin{aligned}\nabla h &= \nabla(\pi_{I_{|\Sigma}}) = (\overline{\nabla}\pi_I)^\top = \epsilon\partial_t^\top\\ &= \epsilon\,\partial_t - \langle N, \partial_t\rangle N.\end{aligned}$$

where $\overline{\nabla}$ denotes the gradient with respect to the metric of the ambient space, and $X^\top$ the tangential component of a vector field $X \in \mathcal{X}(\overline{M})$ in Σ. Now fix $p \in M$, $v \in T_pM$ and let A denote the Weingarten map with respect to N. Write $v = w + \epsilon\langle v, \partial_t\rangle\partial_t$, so that $w \in T_p\overline{M}$ is tangent to the fiber of $\overline{M}$ passing through p. Therefore, by repeated use of the formulas of item (2) of proposition 7.35 of [15], we get

$$\begin{aligned}\overline{\nabla}_v\partial_t &= \overline{\nabla}_w\partial_t + \epsilon\langle v, \partial_t\rangle\overline{\nabla}_{\partial_t}\partial_t = \overline{\nabla}_w\partial_t\\ &= (\log f)'w = (\log f)'(v - \epsilon\langle v, \partial_t\rangle\partial_t),\end{aligned}$$

so that

$$\begin{aligned}\nabla_v\nabla h &= \overline{\nabla}_v\nabla h - \epsilon\langle Av, \nabla h\rangle N\\ &= \overline{\nabla}_v(\epsilon\,\partial_t - \langle N, \partial_t\rangle N) - \epsilon\langle Av, \nabla h\rangle N\\ &= \epsilon(\log f)'w - v(\langle N, \partial_t\rangle)N + \langle N, \partial_t\rangle Av - \epsilon\langle Av, \nabla h\rangle N\\ &= \epsilon(\log f)'w + (\langle Av, \partial_t\rangle - \langle N, \overline{\nabla}_v\partial_t\rangle)N + \langle N, \partial_t\rangle Av - \epsilon\langle Av, \nabla h\rangle N\\ &= \epsilon(\log f)'w + (\langle Av, \partial_t^\top\rangle - \langle N, (\log f)'w\rangle)N + \langle N, \partial_t\rangle Av - \epsilon\langle Av, \nabla h\rangle N\\ &= \epsilon(\log f)'w + \epsilon(\log f)'\langle v, \partial_t\rangle\langle N, \partial_t\rangle N + \langle N, \partial_t\rangle Av\\ &= \epsilon(\log f)'\{v - \langle v, \partial_t\rangle(\epsilon\,\partial_t - \langle N, \partial_t\rangle N)\} + \langle N, \partial_t\rangle Av\\ &= (\log f)'(\epsilon\, v - \epsilon\langle v, \partial_t^\top\rangle\nabla h) + \langle N, \partial_t\rangle Av\\ &= (\log f)'(\epsilon\, v - \langle v, \nabla h\rangle\nabla h) + \langle N, \partial_t\rangle Av\end{aligned}$$

Now, fixing $p \in \Sigma$ and an orthonormal frame $\{e_i\}$ at $T_p\Sigma$, one gets

$$\begin{aligned} \Delta h &= \mathrm{tr}(\nabla^2 h) = \sum_{i=1}^{n} \langle \nabla_{e_i} \nabla h, e_i \rangle \\ &= \sum_{i=1}^{n} \langle (\log f)'(\epsilon\, e_i - \langle e_i, \nabla h\rangle \nabla h) + \langle N, \partial_t \rangle A e_i, e_i \rangle \\ &= (\log f)'\{\epsilon\, n - |\nabla h|^2\} + \langle N, \partial_t\rangle \mathrm{tr}(\mathrm{A}) \\ &= (\log f)'\{\epsilon\, n - |\nabla h|^2\} + \epsilon\, nH\langle N, \partial_t\rangle. \end{aligned}$$

□

Let us consider again a semi-Riemannian warped product $\overline{M}^{n+1} = \epsilon I \times_f M^n$. For $t_0 \in \mathbb{R}$, we orient the fiber $M^n_{t_0} = \{t_0\} \times M^n$ by using the unit normal vector field ∂_t. According to proposition 1 of [12] (see also proposition 1 of [13]), M_{t_0} has constant mean curvature $-\epsilon f'(t_0)/f(t_0)$. We are finally in position to define the objects of our main concern.

Definition 3.3. Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a Riemannian immersion. We say that Σ is a *vertical graph* over the fiber $M^n_{t_0}$ if $\psi(x) = (u(x), x)$ for some smooth function $u : M_{t_0} \to [0, +\infty)$.

Three remarks are in order. First of all, if we let h denote the height function associated to a vertical graph over the fiber M_{t_0}, with corresponding function $u : M_{t_0} \to [0, +\infty)$, then one obviously has $u = h \circ \psi - t_0$. Secondly, in the Lorentz case the condition that ψ is Riemannian in the above definition amounts to $|Du| < 1$, where by Du we mean the gradient of $u \circ \iota$ with respect to the metric of M, where $\iota : M \to M_{t_0}$ is the canonical map (cf. [14], section 4). At last, our applications in the following sections all deal with semi-Riemannian warped products with warping function $f(t) = e^t$. According to the dicussion preceeding the above definition, in this setting all fibers have mean curvature $-\epsilon$, and due to this fact we will assume that our vertical graphs are those over M_0, i.e., such that $u = h \circ \psi \geq 0$. This agreement clarifies our exposition and does not imply in any loss of generality; indeed, changing u by $u + t_0$, all of the arguments to come can be easily adapted to vertical graphs over M_{t_0}.

4. Vertical graphs in the Steady State space

In this section we consider a particular model of Lorentzian warped product, the *Steady State space*, namely, the warped product

$$\mathcal{H}^{n+1} = -\mathbb{R} \times_{e^t} \mathbb{R}^n. \tag{4.1}$$

In Cosmology, this space corresponds to the steady state model of the universe proposed by Bondi, Gold and Hoyle (cf. [8], p. 126).

An alternative description of the Steady State space $\mathcal{H}^{n+1}$ (cf. [14]) can be given as follows. Let $\mathbb{L}^{n+2}$ denote the $(n+2)$-dimensional Lorentz-Minkowski space $(n \geq 2)$, that is, the real vector space $\mathbb{R}^{n+2}$, endowed with the Lorentz metric

$$\langle v, w\rangle = \sum_{i=1}^{n+1} v_i w_i - v_{n+2} w_{n+2},$$

for all $v, w \in \mathbb{R}^{n+2}$. We define the $(n+1)$-dimensional de Sitter space $\mathbb{S}_1^{n+1}$ as the hyperquadric

$$\mathbb{S}_1^{n+1} = \left\{p \in L^{n+2}; \langle p, p\rangle = 1\right\}$$

of $\mathbb{L}^{n+2}$. From the above definition it is easy to show that the metric induced from $\langle\ ,\ \rangle$ turns $\mathbb{S}_1^{n+1}$ into a Lorentz manifold with constant sectional curvature 1. Moreover, for $p \in \mathbb{S}_1^{n+1}$, we have

$$T_p\mathbb{S}_1^{n+1} = \left\{v \in \mathbb{L}^{n+2}; \langle v, p\rangle = 0\right\}.$$

Let $a \in \mathbb{L}^{n+2}$ be a nonzero null vector of the null cone with vertex in the origin, such that $\langle a, e_{n+2}\rangle > 0$, where $e_{n+2} = (0, \ldots, 0, 1)$. It can be shown that the open region

$$\left\{p \in \mathbb{S}_1^{n+1}; \langle p, a\rangle > 0\right\}$$

of the de Sitter space $\mathbb{S}_1^{n+1}$ is isometric to $\mathcal{H}^{n+1}$. Therefore, as a subset of $\mathbb{S}_1^{n+1}$, the boundary of $\mathcal{H}^{n+1}$ is the null hypersurface

$$\left\{p \in \mathbb{S}_1^{n+1}; \langle p, a\rangle = 0\right\}.$$

Back to the warped product model of $\mathcal{H}^{n+1}$, if $\psi : \Sigma^n \to \mathcal{H}^{n+1}$ is a spacelike hypersurface oriented by the timelike unit vector field N such that $\langle N, \partial_t\rangle < 0$, the *hyperbolic angle* θ of ψ is the smooth function $\theta : \psi(\Sigma) \to [0, +\infty)$ such that

$$\cosh\theta = -\langle N, \partial_t\rangle \geq 1. \tag{4.2}$$

In the following result, the right hand side of (4.3) must be interpreted as $+\infty$ when $\cosh\theta = 1$.

Theorem 4.1. *Let $\psi : \Sigma^n \to \mathcal{H}^{n+1}$ be a complete spacelike vertical graph in the $(n+1)$-dimensional Steady State space, with constant mean curvature $H \geq 1$. If*

$$h \leq -\log(\cosh\theta - 1), \tag{4.3}$$

then:

(*a*) *$H = 1$ on Σ.*
(*b*) *The scalar curvature R of Σ is nonnegative and not globally bounded away from zero.*

Proof. Let $g : \Sigma \to \mathbb{R}$ be defined by $g = -e^h - \eta$. It follows easily from (4.2) and the definition of h that $g \geq 0$ on Σ. On the other hand, our hypothesis on the growth of h assures that $g \leq 1$ on Σ.

A straightforward computation gives us $\Delta e^h = e^h\{|\nabla h|^2 + \Delta h\}$. Moreover, since the Riemannian fiber of $\mathcal{H}^{n+1}$ is $\mathbb{R}^n$, by computing the Laplacian of g with the aid of propositions 3.1 and 3.2 we get

$$\begin{aligned}\Delta g &= -\Delta e^h - \Delta\eta \\ &= -e^h\{|\nabla h|^2 + \Delta h\} - \Delta\eta \\ &= ne^h\{1 + H\langle N, \partial_t\rangle\} - \eta|A|^2 - nHe^h.\end{aligned}$$

Now, let S_2 denote the second elementary symmetric function on the eigenvalues of A, and $H_2 = 2S_2/n(n-1)$ denote the mean value of S_2. Elementary algebra gives

$$|A|^2 = n^2H^2 - n(n-1)H_2,$$

which put into the above formula gives, after a little more algebra,

$$\begin{aligned}\Delta g &= n(H-1)\{-e^h - H\eta\} - n(n-1)(H^2-H_2)\eta \\ &\geq n(H-1)g + n(n-1)(H^2-H_2),\end{aligned} \tag{4.4}$$

where for the inequality we used that $-\eta \geq e^h \geq 1$.

(a) Suppose, by contradiction, that $H > 1$. Since $0 \leq g \leq 1$ and (from the Cauchy-Schwarz inequality) $H^2 - H_2 \geq 0$, we get

$$\Delta g \geq n(H-1)g^2.$$

Now let Ric_Σ denote the Ricci curvature of Σ; by applying Gauss' equation, we get the estimate

$$\mathrm{Ric}_\Sigma \geq (n-1) - \frac{n^2H^2}{4}, \tag{4.5}$$

so that we are in position to apply lemma 3 of [5] to conclude that $g \equiv 0$. Thus, $\eta \equiv -e^h$, so that $\langle N, \partial_t\rangle \equiv -1$, i.e., $\psi(\Sigma)$ is a slice of $\mathcal{H}$. However, such a slice has constant mean curvature 1, and we arrive at a contradicion. Thus $H = 1$.

(b) Back to (4.4), we obtain

$$\Delta g \geq n(n-1)(1-H_2) = R \geq 0,$$

where we used Gauss' equation once more to get the last equality, and $H^2 - H_2 \geq 0$ to get the sign for R.

Hence, if there exists $\alpha > 0$ such that $R \geq \alpha$ on Σ, from the above we could derive the inequality

$$\Delta g \geq \alpha g^2,$$

which once more would give us $g \equiv 0$, so that $\psi(\Sigma)$ would also be a slice. However, such a slice is isometric to $\mathbb{R}^n$, thus having scalar curvature $R \equiv 0$. We, therefore, have got another contradiction. □

Remark 4.2. It is easy to see that hypothesis (4.3) on the growth of h is implied by the estimate

$$|\nabla h| \leq e^{-h/2}$$

for the gradient of h, which in turn is taken as a natural one in the literature (see, for instance, corollary 16.6 of [7]).

Remark 4.3. As a consequence of Bonnet-Myers theorem, a complete spacelike hypersurface $\psi : \Sigma^n \to \mathcal{H}^{n+1}$ having (not necessarily constant) mean curvature H, such that $|H| \leq \varrho < 2\sqrt{n-1}/n$ (ϱ constant), has to be compact; in fact, for such a bound on H, equation (4.5) would give us

$$\mathrm{Ric}_\Sigma \geq (n-1) - n^2\rho^2/4 > 0.$$

However, since $\psi(\Sigma)$ is a graph over $\mathbb{R}^n$, it cannot be compact. Therefore, since $2\sqrt{n-1}/n \leq 1$ for $n \geq 2$, in a certain sense it is natural to restrict attention to $H \geq 1$.

As a consequence of the previous result, we have the following Bernstein-type theorem in $\mathcal{H}^3$:

Theorem 4.4. *Let $\psi : \Sigma^2 \to \mathcal{H}^3$ be a complete spacelike vertical graph in the 3-dimensional Steady State space, with constant mean curvature $H \geq 1$. If*

$$h \leq -\log(\cosh\theta - 1),$$

then $\psi(\Sigma)$ is a slice of $\mathcal{H}^3$.

Proof. From the previous result, $H = 1$ on Σ. Now apply the main theorem of [5] and the classification of umbilical hypersurfaces of the de Sitter space, cf. example 1 of [11]. □

We can also apply the result of Proposition 3.2 to prove yet another Bernstein-type theorem for complete surfaces (not necessarily graphs) of the 3−dimensional Steady State space.

Theorem 4.5. *Let $\psi : \Sigma^2 \to \mathcal{H}^3$ be a Riemannian immersion of a complete surface of nonnegative Gaussian curvature K_Σ, with constant mean curvature $H \geq 1$. If*

$$|\nabla h|^2 \leq H^2 - 1, \tag{4.6}$$

then $\psi(\Sigma)$ is a slice of $\mathcal{H}^3$.

Proof. By applying the result of Proposition 3.2, we get

$$\begin{aligned} \triangle e^{-h} &= e^{-h}\left\{|\nabla h|^2 - \triangle h\right\} \\ &= 2e^{-h}\left\{|\nabla h|^2 + 1 + H\langle N, \partial_t\rangle\right\}. \end{aligned}$$

On the other hand, since $|\nabla h|^2 = \langle N, \partial_t\rangle^2 - 1$, hypothesis (4.6) is equivalent to

$$|\nabla h|^2 + 1 + H\langle N, \partial_t\rangle \leq 0,$$

so that the function e^{-h} is a superharmonic positive function on Σ. However, a classical result due to Huber [9] assures that complete surfaces of non-negative Gaussian curvature must be parabolic; therefore, h is constant on Σ, i.e., $\psi(\Sigma)$ is a slice. □

5. Vertical graphs in the Hyperbolic space

In this section, instead of the more commonly used half-space model for the $(n+1)-$dimensional Hyperbolic space, we consider the warped product model

$$\mathbb{H}^{n+1} = \mathbb{R} \times_{e^t} \mathbb{R}^n.$$

An explicit isometry between these two models can be found at [2], from where it can easily be seen that the fibers $M_{t_0} = \{t_0\} \times \mathbb{R}^n$ of the warped product model are precisely the horospheres of $\mathbb{H}^{n+1}$. Moreover, according to the last paragraph of section 3, these have constant mean curvature 1 if we take the orientation given by the unit normal vector field $N = -\partial_t$.

Another useful model for $\mathbb{H}^{n+1}$ is (following the notation of the previous section) the so-called *Lorentz model*, obtained by furnishing the hyperquadric

$$\{p \in \mathbb{L}^{n+2};\, \langle p, p\rangle = -1,\, p_{n+2} > 0\}$$

with the (Riemannian) metric induced by the Lorentz metric of $\mathbb{L}^{n+2}$. In this setting, if $a \in \mathbb{L}^{n+2}$ denotes a fixed null vector as in the beginning of the previous section, a typical horosphere is

$$L_\tau = \{p \in \mathbb{H}^{n+1};\, \langle p, a\rangle = \tau\},$$

where τ is a positive real number. A straightforward computation shows that

$$\xi_p = p + \frac{1}{\tau} a \in \mathcal{H}^{n+1}$$

is a unit normal vector field along L_τ, with respect to which L_τ has mean curvature -1 (cf. [10]). Therefore, any isometry Φ between the warped product and Lorentz models of $\mathbb{H}^{n+1}$ must carry $(\partial_t)_q$ to $\Phi_*(\partial_t) = \xi_{\Phi(q)}$.

If $\psi : \Sigma^n \to \mathbb{H}^{n+1}$ is a vertical graph over $\mathbb{R}^n$, we orient Σ by choosing a unit normal vector field N such that $\eta = \langle N, V\rangle < 0$, and hence $-e^h \leq \eta < 0$. Following the discussion of the previous paragraph, it is natural to consider the *Lorentz Gauss map* of Σ with respect to N as given by

$$\begin{array}{ccc} \Sigma^n & \to & \mathcal{H}^{n+1} \\ p & \mapsto & -\Phi_*(N_p) \end{array}$$

We are finally in position to state and prove, in the Hyperbolic setting, analogues of two of the results of the previous section, starting with theorem 4.1.

Theorem 5.1. *Let Σ be a complete Riemannian manifold with Ricci curvature globally bounded from below, and $\psi : \Sigma^n \to \mathbb{H}^{n+1}$ be a vertical graph in the $(n+1)$-dimensional hyperbolic space, with constant mean curvature $0 \leq H \leq 1$. If*

$$h \leq -\log(1 + \langle N, \partial_t\rangle), \tag{5.1}$$

then:

- (a) *$H = 1$ on Σ.*
- (b) *If the closure of the image of the Lorentz Gauss map of ψ with respect to N is contained in $\mathcal{H}^{n+1}$, then the scalar curvature R of Σ is nonpositive and not globally bounded away from zero.*

Proof. Let $g : \Sigma \to \mathbb{R}$ be defined by $g = e^h + \eta$. The definition of h, together with Cauchy-Schwarz inequality, gives us $g \geq 0$ on Σ; on the other hand, our hypothesis on the growth of h assures that $g \leq 1$ on Σ.

A straightforward computation gives us $\Delta e^h = e^h\{|\nabla h|^2 + \Delta h\}$. Moreover, since the Riemannian fiber of $\mathbb{H}^{n+1}$ is $\mathbb{R}^n$, by computing the Laplacian of g with the aid of propositions 3.1 and 3.2 we get

$$\begin{aligned} \Delta g &= \Delta e^h + \Delta\eta \\ &= e^h\{|\nabla h|^2 + \Delta h\} + \Delta\eta \\ &= ne^h\{1 + H\langle N, \partial_t\rangle\} - \eta|A|^2 - nHe^h. \end{aligned}$$

Now, let S_2 denote the second elementary symmetric function on the eigenvalues of A, and $H_2 = 2S_2/n(n-1)$ denote the mean value of S_2. Elementary algebra gives

$$|A|^2 = n^2H^2 - n(n-1)H_2,$$

which put into the above formula gives, after a little more algebra,

$$\begin{aligned} \Delta g &= n(1-H)\{e^h + H\eta\} - n(n-1)(H^2 - H_2)\eta \\ &= n(1-H)g - n(n-1)(H^2 - H_2)\eta. \end{aligned} \tag{5.2}$$

(a) Suppose, by the sake of contradiction, that $H < 1$ on Σ. Since $0 \leq g \leq 1$, $-\eta > 0$ and $H^2 - H_2 \geq 0$ (from Cauchy-Schwarz inequality), we get

$$\Delta g \geq n(1-H)g^2.$$

Thus, from our hiypothesis on the Ricci curvature of Σ we are in position to apply lemma 3 of [5] to conclude that $g \equiv 0$, which is the same as $\langle N, \partial_t \rangle \equiv -1$. Therefore, $\psi(\Sigma)$ is a horosphere of $\mathbb{H}^{n+1}$. However, such a horosphere has constant mean curvature 1, and we reached a contradicion.

(b) Back to (5.2), we get

$$\Delta g = n(n-1)(H_2 - 1)\eta = R\eta \geq R\langle N, \partial_t \rangle,$$

where Gauss' equation was applied for the last equality and we used that $\eta < 0$ and $H_2 - H^2 \leq 0$ for the last inequality. The condition on the Lorentz Gauss map of Σ amounts to the existence of a real number $\beta > 0$ such that $\langle -N, \partial_t \rangle \geq \beta$ on Σ. Therefore, if there existed a positive real number α such that $R \leq -\alpha$ on Σ, we would get from $0 \leq g \leq 1$ that

$$\Delta g \geq -R\langle -N, \partial_t \rangle \geq \alpha\beta g^2,$$

so that applying lemma 3 of [5] once more would give us $g \equiv 0$. However, horospheres of $\mathbb{H}^{n+1}$ are isometric to $\mathbb{R}^n$, thus having scalar curvature identically 0, which is a contradiction. □

We close this paper with an analogue of theorem 4.5 for the Hyperbolic space.

Theorem 5.2. *Let $\psi : \Sigma^2 \to \mathbb{H}^3$ be a complete vertical graph with nonnegative Gaussian curvature K_Σ and constant mean curvature $\frac{\sqrt{2}}{2} \leq H \leq 1$. If*

$$|\nabla h|^2 \leq 1 - H^2, \tag{5.3}$$

then $\psi(\Sigma)$ is a horosphere of $\mathbb{H}^3$.

Proof. Once more from proposition 3.2, we have

$$\triangle e^{-h} = 2e^{-h}\left\{|\nabla h|^2 - 1 - H\langle N, \partial_t \rangle\right\}.$$

On the other hand, since $|\nabla h|^2 = 1 - \langle N, \partial_t \rangle^2$ and $\langle N, \partial_t \rangle$ does not change sign, hypothesis (5.3) is equivalent to

$$|\nabla h|^2 - 1 - H\langle N, \partial_t \rangle \leq 0,$$

so that e^{-h} is a superharmonic and positive on Σ^2. Hence, as in the proof of theorem 4.5, h is constant on Σ^2, i.e., $\psi(\Sigma)$ is a horosphere. □

Remark 5.3. Since Gauss's equation gives

$$K_\Sigma = 2H^2 - 1 - \frac{1}{2}|A|^2,$$

the assumption $K_\Sigma \geq 0$ forces one to restric attention to the case $H \geq \frac{\sqrt{2}}{2}$.

Remark 5.4. Under the assumption of properness for ψ, a result similar to the above can be found in [2].

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